九十七學年第二學期 PHYS2320 電磁學 第一次期中考(共兩頁)

[Griffiths Ch. 7-8] 2009/04/14, 10:10am–12:00am, 教師：張存續

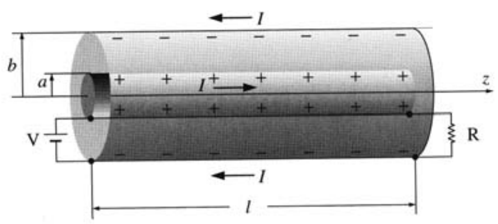
記得寫上學號，班別及姓名等。請依題號順序每頁答一題。

1. (10%, 10%)

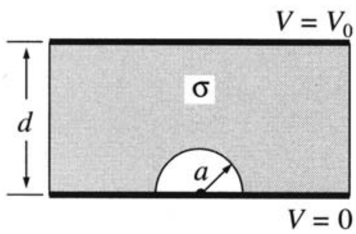
A long coaxial cable of length *l* consists of an inner conductor (radius *a*) and an outer conductor (radius *b*). It is connected to a battery at one end and a resistor at the other, as shown in the figure below. The inner conductor carries a uniform charge per unit length λ, and a steady current *I* to the right; the outer conductor has the opposite charge and current. [Hint: assume the two conductors are held at a potential difference *V*.]

(a) Calculate the **E** and **B** fields.

(b) Calculate the inductance and capacitance per unit length.



2. (20%) Two very large metal plates are held a distance *d* apart, one at potential zero, the other at potential V_0 (as show below). A metal sphere of radius *a* ($a<<d$) is sliced in two, and one hemisphere placed on the ground plate, so the potential is likewise zero. If the region between the plates is filled with weakly conducting material of uniform conductivity σ, what is the resistance *R* between these two plates?



3. (10%, 10%)

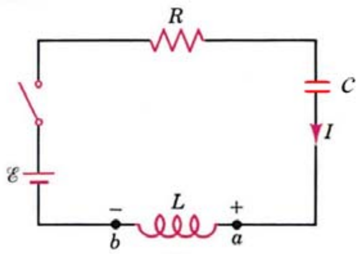
   (a) Write down Maxwell's equations in differential and integral forms.

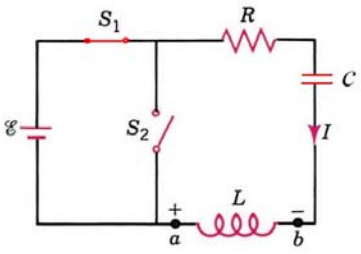
   (b) Prove the conservation of charge using the Maxwell equations.

4. (10%, 10%) A circuit consists of a battery of potential $\mathscr{E}$, an inductor L, a capacitor C, and a resistor R in series.

(a) How does the current rise as a function of time, $I(t)$?

(b) How does the current decay as a function of time, $I(t)$?

Rise 

Decay 

5. (10%, 10%) A point charge q is held a distance d above an infinite grounded conducting plane.

(a) Find the electric field on the grounded plane.

(b) By integrating Maxwell's stress tensor over this plane, determine the force on the charge q.

[Hint: $\mathbf{F} = \oint_S \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a} - \varepsilon_0 \mu_0 \frac{d}{dt} \int_V \mathbf{S} d\tau$ and $T_{ij} \equiv \varepsilon_0 (E_i E_j - \frac{1}{2}\delta_{ij} E^2) + \frac{1}{\mu_0}(B_i B_j - \frac{1}{2}\delta_{ij} B^2)$]

1.(a)

Consider the charge and current reside on the surface of the inner conductor.

We apply Ampere's law and Gauss's law

From Gauss's law

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} (1\%)$$

and the integral form is

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\varepsilon_0} Q_{\text{enc}} \Rightarrow 2\pi sl \left|\mathbf{E}\right| = \frac{1}{\varepsilon_0} \lambda l (2\%)$$

then the electric field is

$$\boxed{\mathbf{E} = \frac{\lambda}{2\pi\varepsilon_0 s} \hat{\mathbf{s}}} (2\%)$$

From Ampere's law

$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ (1%)

and the integral form is

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}} \Rightarrow 2\pi s \left|\mathbf{B}\right| = \mu_0 I_{\text{enc}} \ (2\%)$$

then the magnetic field is

$$\boxed{\mathbf{B} = \frac{\mu_0 I}{2\pi s} \hat{\boldsymbol{\varphi}}} \ (2\%)$$

(b)

The electric energy in the coaxial cable is

$$\frac{\varepsilon_0}{2} \int \left|\mathbf{E}\right|^2 d\tau = \frac{\varepsilon_0}{2} \int_0^l dz \int_0^{2\pi} d\phi \int_a^b s ds \left(\frac{\lambda}{2\pi\varepsilon_0 s} \right)^2 = \frac{\lambda^2 l}{4\pi\varepsilon_0} \ln \frac{b}{a} \ (3\%)$$

this magnitude is equal to $\frac{1}{2} CV^2$ (1%)

and $V = \int_a^b \mathbf{E} \cdot d\mathbf{s} = \int_a^b \frac{\lambda}{2\pi\varepsilon_0 s} ds = \frac{\lambda}{2\pi\varepsilon_0} \ln \frac{b}{a}$

We obtain $\boxed{\frac{C}{l} = \frac{2\pi\varepsilon_0}{\ln b/a}}$ (1%)

Similarly,

$$\frac{1}{2} LI^2 = \frac{1}{2\mu_0} \int \left|\mathbf{B}\right|^2 d\tau \ (1\%)$$

$$\frac{1}{2\mu_0} \int \left|\mathbf{B}\right|^2 d\tau = \frac{1}{2\mu_0} \int_0^l dz \int_0^{2\pi} d\phi \int_a^b s ds \left(\frac{\mu_0 I}{2\pi s} \right)^2 = \frac{\mu_0 I^2 l}{4\pi} \ln \frac{b}{a} \ (3\%)$$

and hence, we obtain $\boxed{\frac{L}{l} = \frac{\mu_0}{2\pi} \ln \frac{b}{a}}$ (1%)

2.

Boundary conditions $\begin{cases} \text{(i) } V_{lower\ plate}(r,\theta=\pi/2)=0 \\ \text{(ii) } V_{sphere}(a,\theta)=0 \\ \text{(iii) } V_{upper\ plate}(r\cos\theta=d)=V_0 \end{cases}$

General solution $V(r,\theta)=\sum_{\ell=0}^{\infty}(A_\ell r^\ell + B_\ell r^{-(\ell+1)})P_\ell(\cos\theta)$

B.C. (i): $V(r,\pi/2)=\sum_{\ell=1,3,5...}^{\infty}(A_\ell r^\ell + B_\ell r^{-(\ell+1)})P_\ell(\cos\theta)=0,$ since $P_\ell(0)=0$

B. C. (ii) $V(a,\theta)=\sum_{\ell=1,3,5...}^{\infty}(A_\ell a^\ell + B_\ell a^{-(\ell+1)})P_\ell(\cos\theta)=0, \ \Rightarrow\ B_\ell=-a^{2\ell+1}A_\ell$

B. C. (iii) $V(r\cos\theta=d)=\sum_{\ell=1,3,5...}^{\infty}A_\ell(d^\ell-\frac{a^{2\ell+1}}{d^{\ell+1}})P_\ell(1)=V_0,$ since $d>>a$ and $P_\ell(1)=1$

$A_1=\frac{V_0}{d},\ A_\ell=0$ for $\ell=3,5,7,...$

$V(r,\theta)=\frac{V_0}{d}(r-\frac{a^3}{r^2})\cos\theta,$ {Ex. 3.8, a metal sphere splaced in a uniform E-field. }

$\mathbf{E}=-\nabla V=-\frac{V_0}{d}(1+\frac{2a^3}{r^3})\cos\theta\hat{\mathbf{r}}+\frac{V_0}{d}(1-\frac{a^3}{r^3})\sin\theta\hat{\boldsymbol{\theta}}$ {Check the field.}

$$I=\int_s \mathbf{J}\cdot d\mathbf{a}=\sigma\int_s \mathbf{E}\cdot d\mathbf{a}=\sigma\int_{sphere}\mathbf{E}\cdot d\mathbf{a}+\sigma\int_{plane}\mathbf{E}\cdot d\mathbf{a}$$

$$=\sigma\int_0^{\pi/2}-\frac{V_0}{d}(1+\frac{2a^3}{a^3})\cos\theta\cdot 2\pi a^2\sin\theta d\theta+\sigma\int_{r=a}^{r=R}\frac{V_0}{d}(1-\frac{a^3}{r^3})\cdot 2\pi r dr$$

$$=-\sigma\frac{3\pi a^2 V_0}{d}\int_0^{\pi/2}\sin 2\theta d\theta+\sigma\frac{2\pi V_0}{d}\int_{r=a}^{r=R}(1-\frac{a^3}{r^3})\cdot r dr$$

$$=\underbrace{-\sigma\frac{3\pi a^2 V_0}{d}}_{\text{from the hemisphere}}+\underbrace{\sigma\frac{V_0}{d}(\pi(R^2-a^2)+2\pi(\frac{a^3}{R}-a^2)}_{\text{from the plane}},\ A=\pi R^2$$

$$=\sigma(\frac{A}{d}-\frac{6\pi a^2}{d})V_0$$

So the resistivity is $R=\frac{d}{\sigma(A-6\pi a^2)}$

3.(a)differential form: $\nabla\cdot\vec{E}=\rho/\varepsilon_o$ $\qquad$ $\nabla\cdot B=0$

$$\nabla\times\vec{E}=-\frac{\partial B}{\partial t} \qquad \nabla\times\vec{B}=\mu_o\vec{J}+\mu_o\varepsilon_o\frac{\partial E}{\partial t}$$

integral form: $\oint\vec{E}\cdot d\vec{a}=\frac{Q}{\varepsilon_o}$ $\qquad$ $\oint\vec{B}\cdot d\vec{a}=0$

$$\oint\vec{E}\cdot d\vec{l}=-\frac{\partial}{\partial t}\int\vec{B}\cdot d\vec{a} \qquad \oint\vec{B}\cdot d\vec{l}=\mu_o\vec{I}+\mu_o\varepsilon_o\frac{\partial}{\partial t}\int\vec{E}\cdot d\vec{a}$$

（每個式子 1 分）

(b) $Q=\int_v\rho d\tau$ $\qquad$ $\frac{dQ}{dt}=-\int_s\vec{J}\cdot d\vec{a}$

$\int_v\frac{\partial\rho}{\partial t}d\tau=-\int\nabla\cdot\vec{J}d\tau$ $\qquad$ $\Rightarrow\frac{\partial\rho}{\partial t}=-\nabla\cdot\vec{J}$（只寫結果 3 分）

or $\quad\nabla\cdot\nabla\times\vec{B}=\mu_o\nabla\cdot\vec{J}+\mu_o\varepsilon_o\frac{\partial\nabla\cdot E}{\partial t}=0$

$$\nabla\cdot\vec{J}=-\varepsilon_o\frac{\partial\rho}{\varepsilon_o\partial t} \qquad \Rightarrow\frac{\partial\rho}{\partial t}=-\nabla\cdot\vec{J}$$

4.(a) $\varepsilon-L\frac{dI}{dt}-\frac{Q}{c}=IR$ （4 分） $\qquad$ 又 $I=\frac{dQ}{dt}$ （2 分）

$$\Rightarrow\frac{d^2Q}{dt^2}+\frac{R}{L}\frac{dQ}{dt}+\frac{Q}{LC}=\frac{\varepsilon}{L}$$

For homogeneous sol:

令 $Q=e^{\lambda t}$

$\lambda^2+\frac{R}{L}\lambda+\frac{1}{LC}=0$ $\qquad$ $\lambda=\frac{-\frac{R}{L}\pm\sqrt{\left(\frac{R}{L}\right)^2-\frac{4}{LC}}}{2}$ (2 分)

$Q_h=Ae^{\lambda_1 t}+Be^{\lambda_2 t}$

For particular sol:

$Q_p=c\,\varepsilon$ (1 分)

代入邊界條件 $Q(0)=0$ $\qquad$ $Q=Q_h+Q_p$（1 分）

(b) $\Rightarrow\frac{d^2Q}{dt^2}+\frac{R}{L}\frac{dQ}{dt}+\frac{Q}{LC}=0$ （6 分）

照上題 homogeneous 的解法求出 λ（2 分）

列出通解 （1 分）

代入邊界條件 Q（0）=c ε (1 分)

5. (a)

The potential of a point charge is given by

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{|\mathbf{r}-\mathbf{r}'|} \quad (2\%)$$

Using the method of image, we can obtain the potential

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{|\mathbf{r}-\mathbf{r}'|} + \frac{1}{4\pi\varepsilon_0}\frac{q'}{|\mathbf{r}-\mathbf{r}''|} \quad (2\%)$$

where $q' = -q$, $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$, $\mathbf{r}' = d\hat{\mathbf{z}}$, $\mathbf{r}'' = -d\hat{\mathbf{z}}$

therefore $V(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+d)^2}}\right]$ (3%)

and the electric field on the grounded plane is

$$\mathbf{E}(z=0) = -\nabla V(z=0) = \boxed{-\frac{q}{4\pi\varepsilon_0}\frac{2d}{\left(x^2+y^2+d^2\right)^{3/2}}\hat{\mathbf{z}}} \quad (3\%)$$

(b)

Since there has no magnetic field, the poynting vector vanishes. (1%)

Hence the force is only due to Maxwell's stress tensor, and the x component and y component of electric field is equal to zero

$T_{ij} = 0$ for $i \neq j$

$T_{33} = \frac{1}{2}E^2$ where $E = |\mathbf{E}|$ (4%)

Then the force on the plane (its normal vector is $\hat{\mathbf{z}}$) is

$$\mathbf{F} = \oint \overleftrightarrow{\mathbf{T}} \cdot \hat{\mathbf{z}}da = \oint (T_{13}\hat{\mathbf{x}} + T_{23}\hat{\mathbf{y}} + T_{33}\hat{\mathbf{z}})da = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \varepsilon_0 \left(\frac{1}{2}\right)\left[\frac{q}{4\pi\varepsilon_0}\frac{2d}{\left(x^2+y^2+d^2\right)^{3/2}}\right]^2 \hat{\mathbf{z}} \quad (2\%)$$

$$= \frac{\varepsilon_0}{2}\left(\frac{q}{4\pi\varepsilon_0}\right)^2 4d^2 \int_0^{2\pi} d\phi \int_0^{\infty} r dr \frac{1}{\left(r^2+d^2\right)^3}\hat{\mathbf{z}}$$

$$= \frac{q^2}{16\pi\varepsilon_0 d^2}\hat{\mathbf{z}}$$

Because the plane and the charge attract each other, the force on the charge is $\boxed{-\frac{q^2}{16\pi\varepsilon_0 d^2}\hat{\mathbf{z}}}$ (3%)